Control problems and HJB equations

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Outline

1. Controlled systems. Optimal control problems
2. Value functions. HJB equations
3. HJB equations. Viscosity theory
4. Planing Motion, reachability analysis
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1. Controlled systems. Optimal control problems
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Trajectory optimization problem for a space launcher

Aim

Maximize the payload $m_0$ to be steered from the Earth (Kourou) to a prescribed Orbit (SSO, GEO, ...).

The physical model involves $6+1$ state variables, the position $\vec{X}$ of the launcher in the 3D space, its velocity $\vec{V}$ and its mass $M$:

$$ y := (X, V, M). $$

The forces acting on the rocket are: Gravity $M\vec{g}$, Thrust $\vec{F}_T$, Drag $\vec{F}_D$, and Coriolis forces.

Newton’s Law:

$$ \frac{d\vec{X}}{dt} = \vec{V}, $$

$$ \frac{d\vec{V}}{dt} = \vec{g} + \frac{\vec{F}_T}{M} + \frac{\vec{F}_D}{M} - 2\vec{\Omega} \wedge \vec{V} - \vec{\Omega} \wedge (\vec{\Omega} \wedge \vec{X}), $$

The launcher is controlled by means of:

- **launch parameters** $\Pi = (\psi, \omega) \in \mathbb{R}^2$
- **incidence and sideslip angles** $\alpha(\cdot), \delta(\cdot)$

Physical state-constraints (intermediate times, end-point, along the time interval)
A simplified problem

Example (Goddard problem)

\[
\dot{r}(s) = v(s)
\]
\[
\dot{v}(s) = \frac{1}{m} (F_T u(s) - F_D(r, v)) - \frac{1}{r(s)^2}
\]
\[
\dot{m}(s) = -\beta F_T u(s)
\]
\[
r(0) = 1, \ v(0) = 0, \ m(0) = 1
\]

- The ratio \(u(t)\) is subject to the following constraint: \(0 \leq u(t) \leq 1\).
- The rocket’s mass satisfies the final constraint: \(r(T) \geq r^*\).

The optimal control problem is the following (for \(T > 0\)):

\[
\max_{u(t) \in [0, 1]} m(T) \quad \text{subject to} \quad r(T) \geq r^*.
\]
Figure: Optimal control laws for different final times $T = t_f$. 
Major development since the 50s

- Pontryagin’s principle: necessary optimality conditions (Euler-Lagrange and Weierstrass conditions)

- Bellman’s principle: dynamic programming principle (Verification theorem)

- Important breakthroughs in the 80s: nonsmooth analysis, viscosity theory.
Mathematical formulation of the Control problem

- For a given non-empty compact subset $A$ of $\mathbb{R}^k$, define the set of admissible controls as:

$$A := \left\{ \alpha : (0, +\infty) \to \mathbb{R}^k, \text{ measurable, } \alpha(t) \in A \text{ a.e } \right\}.$$  

- Consider the following control system:

$$\begin{align*}
\dot{y}(s) &:= f(y(s), \alpha(s)), \quad \text{a.e } s \in [0, t], \\
 y(0) &:= x,
\end{align*}$$  

where $f : \mathbb{R}^d \times A \to \mathbb{R}^d$ is continuous, and Lipschitz continuous w.r.t $y$.

- Define the set of trajectories:

$$S_{[0,t]}(x) := \{ y_x^\alpha \in W^{1,1}(0, t; \mathbb{R}^d), \ y_x^\alpha \text{ satisfies } (1) \text{ for some } \alpha \in A \},$$

The multi-application: $x \rightsquigarrow S_{[0,t]}(x)$ is Lipschitz continuous; i.e.,

$$\exists L > 0, S_{[0,t]}(x) \subset S_{[0,t]}(z) + L |x - z|_{\mathcal{B}}$$  \quad \forall x, z \in \mathbb{R}^d.$$
Set of constrained trajectories

Assume that $f(x, A) := \{f(x, a), a \in A\}$ is a convex set. Then, by Filippov’s theorem, the set of trajectories $S_{[0,t]}(x)$ is a compact set of $W^{1,1}([0, t])$ endowed with the $C^0$-topology.
Set of constrained trajectories

- Assume that \( f(x, A) := \{ f(x, a), \ a \in A \} \) is a convex set. Then, by Filippov’s theorem, the set of trajectories \( S_{[0,t]}(x) \) is a compact set of \( W^{1,1}([0,t]) \) endowed with the \( C^0 \)-topology.

- The set of feasible trajectories:

\[
S_{g_{[0,t]}}^g(x) := \{ y \in S_{[0,t]}(x) \mid g(y(s)) \leq 0 \ \forall s \in [0,t] \}
\]

is a compact subset of \( W^{1,1}([0,t]) \) ... when it is non-empty!
Set of constrained trajectories

Assume that $f(x, A) := \{ f(x, a), \ a \in A \}$ is a convex set. Then, by Filippov’s theorem, the set of trajectories $S_{[0,t]}(x)$ is a compact set of $W^{1,1}([0, t])$ endowed with the $C^0$-topology.

The set of feasible trajectories:

$$S^g_{[0,t]}(x) := \{ y \in S_{[0,t]}(x) \mid g(y(s)) \leq 0 \ \forall s \in [0, t] \}$$

is a compact subset of $W^{1,1}([0, t])$ ... when it is non-empty!

Inward pointing (IP) condition: Assume $g$ smooth and

$$\exists \beta > 0, \ \forall x \ \text{s.t.} \ g(x) = 0, \ \min_{a \in A} f(x, A) \cdot \nabla g(x) < -\beta.$$ 

Then, for $x \in \mathcal{K}$, $S^g_{[0,t]}(x) \neq \emptyset$, and $x \mapsto S^g_{[0,t]}(x)$ is Lipschitz.

Ref: Arutyunov’84, Soner’86, Rampazzo-Vinter’99, Vinter-Frankowska’00, Clarke-Rifford-Stern’02, Hermosilla-Vinter-HZ’18 ...
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Now, consider the following control problems:

➤ Mayer’s problem:

\[ V(x, t) = \inf_{y_x \in S_{[0,t]}(x)} \Phi(y_x(t)) \]

➤ Time minimum problem (\( \mathcal{C} \) closed set in \( \mathbb{R}^d \)):

\[ \mathcal{T}(x) = \inf \left\{ t; \ y_x(t) \in \mathcal{C}, \ y_x \in S_{[0,t]}(x) \right\} \]

➤ Supremum cost:

\[ V^\infty(x, t) = \inf_{y_x \in S_{[0,t]}(x)} \Phi(y_x(t)) \bigvee_{\theta \in [0,t]} \sup g(y_x(\theta)) \]
Assume that \( f(x, A) := \{ f(x, a), \ a \in A \} \) is a convex set. Assume that the function \( g \) is Lipschitz continuous.

- If \( \phi : \mathbb{R}^d \to \mathbb{R} \) is Lipschitz continuous, then \( V \) and \( V^\infty \) are Lipschitz continuous on \( \mathbb{R}^d \times [0, T] \) for every \( T > 0 \).
Assume that \( f(x, A) := \{ f(x, a), \ a \in A \} \) is a convex set.
Assume that the function \( g \) is Lipschitz continuous.

If \( \Phi : \mathbb{R}^d \to \mathbb{R} \) is Lipschitz continuous, then \( V \) and \( V^\infty \) are Lipschitz continuous on \( \mathbb{R}^d \times [0, T] \) for every \( T > 0 \).

\[
|V(x, t) - V(x', t)| \leq \sup_{y_x \in S_{[0,t]}(x)} |\Phi(y_x(t)) - \Phi(y_x'(t))| \\
\leq L_\Phi \sup_{y_x \in S_{[0,t]}(x)} |y_x(t) - y_x'(t)| \\
\leq L_\Phi e^{L_\Phi T} |x - x'|.
\]
Assume that $f(x, A) := \{ f(x, a), \ a \in A \}$ is a convex set.
Assume that the function $g$ is Lipschitz continuous.

- If $\Phi : \mathbb{R}^d \to \mathbb{R}$ is Lipschitz continuous, then $V$ and $V^\infty$ are Lipschitz continuous on $\mathbb{R}^d \times [0, T]$ for every $T > 0$.

- If $\Phi : \mathbb{R}^d \to \mathbb{R}$ is lsc, then $V$ and $V^\infty$ are lsc.
Assume that \( f(x, A) := \{ f(x, a), \ a \in A \} \) is a convex set.
Assume that the function \( g \) is Lipschitz continuous.

- If \( \Phi : \mathbb{R}^d \to \mathbb{R} \) is Lipschitz continuous, then \( V \) and \( V^\infty \) are Lipschitz continuous on \( \mathbb{R}^d \times [0, T] \) for every \( T > 0 \).
- If \( \Phi : \mathbb{R}^d \to \mathbb{R} \) is lsc, then \( V \) and \( V^\infty \) are lsc.
- When the target \( C \) is closed, the minimum time function \( T \) is lsc.
Dynamic programming principle: HJB equation

Mayer Problem

\[ V(x, t) = \min_{y_x \in S_{[0, h]}(x)} V(y_x(h), t - h) \quad h \in (0, t), \]
\[ V(x, 0) = \Phi(x) \]
Dynamic programming principle: HJB equation

Mayer Problem

\[ V(x, t) = \min_{y_x \in S_{[0, h]}(x)} V(y_x(h), t - h) \quad h \in (0, t), \]
\[ V(x, 0) = \Phi(x) \]

Minimum time problem:

\[ T(x) = \min_{y_x \in S_{[0, h]}(x)} T(y_x(h)) + h \quad h < T(x), \quad x \notin C, \]
\[ T(x) = 0 \quad x \in C; \]
## Dynamic programming principle: HJB equation

### Mayer Problem

\[
V(x, t) = \min_{y_x \in S_{[0, h]}(x)} V(y_x(h), t - h) \quad h \in (0, t),
\]

\[
V(x, 0) = \Phi(x)
\]

### Minimum time problem:

\[
T(x) = \min_{y_x \in S_{[0, h]}(x)} T(y_x(h)) + h \quad h < T(x), \ x \notin C,
\]

\[
T(x) = 0 \quad x \in C;
\]

### Supremum cost

\[
V^\infty(x, t) = \min_{y_x \in S_{[0, h]}(x)} V^\infty(y_x(h), t - h) \vee \sup_{\theta \in [0, h]} g(y_x(\theta))
\]

\[
V^\infty(x, 0) = \Phi(x) \vee g(x);
\]
$$V(x, t) = \min_{y_x \in S_{[0,t]}(x)} \Phi(y_x(t))$$

$$V(x, t) = \min_{y_x \in S_{[0,t]}(x)} V(y_x(h), t - h) \quad h \in (0, t).$$

$$V(x, 0) = \Phi(x)$$

- **Suboptimality:**
  $$\forall y_x \in S_{[0,t]}(x), \quad s \mapsto V(y_x(s), t - s) \text{ is increasing},$$

- **Superoptimality**
  $$\exists y_x^* \in S_{[0,t]}(x), \quad s \mapsto V(y_x^*(s), t - s) \text{ is constant}$$
Next, we derive the Hamilton-Jacobi-Bellman equation (HJB), which is an infinitesimal version of the DPP.

If the value function is $C^1$ in a neighborhood of $(x, t)$, then

1. $\partial_t V(x, t) + \mathcal{H}(x, D_x V(x, t)) = 0, \quad x \in \mathbb{R}^d, \ t > 0$;

2. $\mathcal{H}(x, DT(x)) = 1, \quad x \notin \mathcal{C}, \ T(x) < +\infty$;

3. $\min(\partial_t V^\infty(x, t) + \mathcal{H}(x, D V^\infty(x, t)), V^\infty(x, t) - g(x)) = 0, \quad x \in \mathbb{R}^d, \ t > 0$;

where

$$\mathcal{H}(x, q) := \max_{a \in \mathcal{A}} (-f(x, a) \cdot q)$$
Next, we derive the Hamilton-Jacobi-Bellman equation (HJB), which is an infinitesimal version of the DPP.

If the value function is $C^1$ in a neighborhood of $(x, t)$, then

\begin{itemize}
  \item $\partial_t V(x, t) + \mathcal{H}(x, D_x V(x, t)) = 0, \quad x \in \mathbb{R}^d, \; t > 0$;  
  $V(x, 0) = \Phi(x)$ \quad \text{Time-dependent HJB equation}
  \item $\mathcal{H}(x, DT(x)) = 1, \quad x \not\in C, \; T(x) < +\infty$;  
  $T(x) = 0$ on $C$ \quad \text{Steady HJB equation}
  \item $\min(\partial_t V^\infty(x, t) + \mathcal{H}(x, DV^\infty(x, t)), V^\infty(x, t) - g(x)) = 0$,  
  $V^\infty(x, 0) = \Phi(x) \lor g(x)$ \quad \text{HJB-VI inequation}
\end{itemize}

where

$$\mathcal{H}(x, q) := \max_a (-f(x, a) \cdot q)$$
Verification Theorem

Let \( u \in C^1_b(\mathbb{R}^d \times [0, T]) \). We say that \( u \) is a classical verification function if it satisfies

\[
\partial_t u(x, t) + \mathcal{H}(x, D_x u(x, t)) \leq 0 \quad x \in \mathbb{R}^d, t \in [0, T],
\]

\[ u(x, 0) = \Phi(x); \]

with the Hamiltonian function

\[
\mathcal{H}(x, q) = \max_{a \in A} (-f(x, a) \cdot q).
\]
Verification Theorem

Let \( u \in C^1_b(\mathbb{R}^d \times [0, T]) \). We say that \( u \) is a classical verification function if it satisfies

\[
\partial_t u(x, t) + \mathcal{H}(x, D_x u(x, t)) \leq 0 \quad x \in \mathbb{R}^d, \ t \in [0, T],
\]

\[
u(x, 0) = \Phi(x);
\]

Proposition

Let \( y^*_x \in S_{[0, T]}(x) \) be a feasible arc. Assume that there exists a classical verification function \( u \), s.t.

\[
u(x, T) = \Phi(y^*_x(T)),
\]

then \( y^*_x \) is optimal for \( x \) on \([0, T]\).
Proof.

▷ We take any $y_x \in S_{[0, T]}(x)$, we have:

$$\frac{d}{ds}[u(y_x(s), T - s)] = -\partial_t u(y_x(s), T - s) + \dot{y}_x(s) \cdot Du(y_x(s), T - s) \geq 0.$$  

Which means that $s \mapsto u(y_x(s), T - s)$ is increasing:

$$u(x, T) \leq u(y_x(T), 0) = \Phi(y_x(T)).$$

Therefore, $u(x, T) \leq V(x, T)$.

▷ $u(x, T) = \Phi(y^*_x(T)) \geq V(x, T)$.

▷ Therefore $u(x, T) = \Phi(y^*_x(T)) = V(x, T)$, and $y^*_x$ is optimal.
Proposition

Let $y_x^* \in S_{[0, T]}(x)$ be a feasible arc, and let $a^*(\cdot) \in A$ be the corresponding control strategy:

$$\dot{y}_x^*(t) = f(y_x^*(t), a^*(t)) \text{ a.e in } [0, T].$$

Assume that there exists a classical verification function $u$, s.t.

$$\partial_t u(y_x^*(t), t) - f(y_x^*(t), a^*(t)).Du(y_x^*(t), t) = 0 \text{ for a.e } t \in [0, T],$$

then $y_x^*$ is optimal for $x$ on $[0, T]$. 
Proof.

- We know that $s \mapsto u(y_x(s), T - s)$ is increasing:
  
  $$u(x, T) \leq u(y_x(T), 0) = \Phi(y_x(T)).$$

  Therefore, $u(x, T) \leq V(x, T)$.

- $u(y_x^*(s), T - s) = u(x, T) = \text{Cst} = u(y_x^*(T), 0) = \Phi(y_x^*(T)) \geq V(x, T)$.

- Therefore $u(x, T) = \Phi(y_x^*(T)) = V(x, T)$, and $y_x^*$ is optimal.
If we take the value function $V$ itself as a verification function (i.e., if $V$ is smooth), then a necessary and sufficient condition of optimality is:

$$-f(y_x^*(t), a^*(t)) \cdot DV(y_x^*(t), t) = \max_{a \in A} (-f(y_x^*(t), a) \cdot DV(y_x^*(t), t))$$

$$= \mathcal{H}(y_x^*(t), DV(y_x^*(t), t))$$

for a.e $0 < t < T$. 
Under the (irrealistic!) assumption that $V$ is smooth, we consider the multivalued function:

$$
\psi(x, t) := \text{argmax}_{a \in A} (-f(x, a) \cdot DV(x, t)).
$$

Let $y_x^* \in S_{[0, t]}(x)$. $y_x^*$ is an optimal trajectory for $x$ on $[0, T]$ if and only if $y_x^*(0) = x$, and

$$
\dot{y}_x^*(t) \in f(y_x^*(t), \psi(y_x^*(t), t)) \quad t \in (0, T).
$$
Van der Pol Problem:

\[
\begin{align*}
\dot{y}_1(t) &= y_2 \\
\dot{y}_2(t) &= -y_1 + y_2(1 - y_1^2) + a(t) \\
a(t) &\in [-1, 1]
\end{align*}
\]

The final cost function:

\[
\Phi(y) := \|y\| - r_0
\]

The HJB equation to be solved for this problem is:

\[
-\partial_t \vartheta(x, t) + \left( x_1 + x_2(1 - x_1^2) \right) \cdot D_x \vartheta(x, t) + |\partial_{x_2} \vartheta(x, t)| = 0
\]

\[
\vartheta(x, 0) = \Phi(x) = \|x\| - r_0.
\]
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Let $\Omega \subset \mathbb{R}^M$, and $\mathcal{F} : \Omega \times \mathbb{R} \times \mathbb{R}^M \to \mathbb{R}$ be continuous. Consider the nonlinear PDE:

$$\mathcal{F}(x, u(x), Du(x)) = 0 \quad \text{in } \Omega,$$

(2)

**Definition.**

- A function $u \in USC(\Omega)$ is a viscosity subsolution of (2) if for any $\varphi \in C^1(\Omega)$ and any local maximum point $x_0$ of $u - \varphi$,

$$\mathcal{F}(x, u(x), D\varphi(x)) \leq 0.$$

- A function $u \in LSC(\Omega)$ is a viscosity supersolution of (2) if for any $\varphi \in C^1(\Omega)$ and any local minimum point $x_0$ of $u - \varphi$,

$$\mathcal{F}(x, u(x), D\varphi(x)) \geq 0.$$

- A viscosity solution of (2) is a subsolution which is also supersolution.
Let $\Omega \subset \mathbb{R}^M$, and $\mathcal{F}: \Omega \times \mathbb{R} \times \mathbb{R}^M \to \mathbb{R}$ be continuous. Consider the nonlinear PDE:

$$\mathcal{F}(x, u(x), Du(x)) = 0 \quad \text{in } \Omega,$$

(2)

**Definition.**

- A function $u \in USC(\Omega)$ is a viscosity subsolution of (2) if for every $q \in D^+ u(x)$, we have:

$$\mathcal{F}(x, u(x), q) \leq 0.$$

- A function $u \in LSC(\Omega)$ is a viscosity supersolution of (2) if for every $q \in D^- u(x)$, we have:

$$\mathcal{F}(x, u(x), q) \geq 0.$$

$q \in D^+ u(x) \iff u(y) \leq u(x) + q \cdot (y - x) + o(|y - x|)$,

$q \in D^- u(x) \iff u(y) \geq u(x) + q \cdot (y - x) + o(|y - x|).$
Theorem

Assume that the value is continuous. Then it is a viscosity solution of the corresponding HJB equation.

Proof for the Minimum time problem. Let $\varphi \in C^1$ is such that $\mathcal{T} - \varphi$ has a local maximum at $x \notin \mathcal{C}$, then for any $y_x \in S_{[0,s]}(x)$ we have:

$$\varphi(x) - \varphi(y_x(s)) \leq \mathcal{T}(x) - \mathcal{T}(y_x(s)) \leq s$$

for small $s$. Therefore,

$$\sup_{p \in F(x)} (-p \cdot D\varphi(x)) = H(x, D\varphi) \leq 1,$$

so $\mathcal{T}$ is a subsolution. With similar arguments, we prove that $\mathcal{T}$ is a supersolution as well.
The value function satisfies a Hamilton-Jacobi-Bellman equation (HJB) in a viscosity sense.

1. \[ \partial_t V(x, t) + H(x, D_x V(x, t)) = 0, \quad x \in \mathbb{R}^d, \ t > 0; \]
   
2. \[ H(x, D_T(x)) = 1, \quad x \not\in \mathcal{C}, \ T(x) < +\infty; \]
   
3. \[ \min(\partial_t V^\infty(x, t) + H(x, D V^\infty(x, t)), V^\infty(x, t) - g(x)) = 0, \]
   \[ x \in \mathbb{R}^d, \ t > 0; \]

where \( H(x, q) := \max_{p \in F(x)} (-p \cdot q). \)
The value function satisfies a Hamilton-Jacobi-Bellman equation (HJB) in a viscosity sense.

\[ \partial_t V(x, t) + H(x, D_x V(x, t)) = 0, \quad x \in \mathbb{R}^d, \ t > 0; \]
\[ V(x, 0) = \Phi(x) \]

**Time-dependent HJB equation**

\[ H(x, D\mathcal{T}(x)) = 1, \quad x \notin \mathcal{C}, \ \mathcal{T}(x) < +\infty; \]
\[ \mathcal{T}(x) = 0 \text{ on } \mathcal{C} \]

**Steady HJB equation**

\[ \min(\partial_t V^\infty(x, t) + H(x, D\cal{V}^\infty(x, t)), \ V^\infty(x, t) - g(x)) = 0, \]
\[ V^\infty(x, 0) = \Phi(x) \lor g(x) \]

**HJB-VI inequation**

where \( H(x, q) := \max_{p \in F(x)}(-p.q) \).
Assume the value function $V$ (resp. $V^\infty$) is continuous and bounded. Then it is the unique bounded and continuous viscosity solution to the corresponding HJB equation.

The proof of this result is an immediate consequence of the comparison principle for Cauchy problem (see e.g. Crandall-Lions, Crandall-Evans-Lions, Barles, Bardi-Capuzzo Dolcetta, Ishii, ...)
The boundedness assumption on $V$ and $V^\infty$ is not restrictive. It is verified whenever $\Phi$ and $g$ are bounded.

The continuity of $V$ and $V^\infty$ holds whenever $\Phi$ and $g$ are continuous.

The uniqueness results for the HJB equation associated to the minimal time problem is not a trivial task.
Assume $C$ is a closure of smooth domain, and let $\eta_x$ be the normal to $C$. The minimal time function is Lipschitz continuous if and only if

$$\min_{a \in A} f(x, a) \cdot \eta_x < 0, \quad \forall x \in \partial C.$$ 

The continuity of $T$ requires a controllability assumption of the system around the target. This important property is not satisfied in several examples.
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Reachable (or Attainable) set

The reachable set $\mathcal{R}(x; t)$ from $x$ at time $t$ is the set of all points of the form $y_x(\tau)$, where $y_x \in S_{[0,t]}(x)$:

$$\mathcal{R}(x; t) := \{y_x(\tau) \mid y_x \in S_{[0,t]}(x), \tau \in [0, t]\}.$$

The reachable set from $X$ is defined by:

$$\mathcal{R}_X(t) := \bigcup_{x \in X} \mathcal{R}(x; t).$$
Some properties of the reachable sets

- If \( f(x, A) \) is a non-empty compact set (not necessarily convex) then \( R_C(t) \) is not necessarily closed.

- Assume \( f(x, A) \) is a non-empty compact and convex set, then for every \( t \geq 0 \), \( R_C(t) \) is a closed set.

In all the sequel, we will assume that \( f(x, A) \) is a non-empty compact and convex set.
Let $C$ be a closed target set (in our examples, $C$ is safe)

**Capture Basin (or Backward reachable set)**

- The Capture Basin $R^C_{\#t}$, at time $t$, is the set of all initial positions $x$ from which a trajectory $y_x \in S_{[0,t]}(x)$ can reach the target $C$. 
Let $C$ be a closed target set (in our examples, $C$ is safe).

**Capture Basin (or Backward reachable set)**

- The Capture Basin $\mathcal{R}_t^C$, before time $t$, is the set of all initial positions $x$ from which a trajectory $y_x \in S_{[0,t]}(x)$ can reach the target $C$ before $t$.

$$
\mathcal{R}_t^C := \{ x \in \mathbb{R}^d, \exists \tau \in [0, t], \exists y_x \in S_{[0,\tau]}(x), \ y_x(\tau) \in C \} 
$$
Let $C$ be a closed target set (in our examples, $C$ is safe)

**Capture Basin (or Backward reachable set)**

- The Capture Basin $\mathcal{R}_t^C$, before time $t$, is the set of all initial positions $x$ from which a trajectory $y_x \in S_{[0,t]}(x)$ can reach the target $C$ before $t$.

$$\mathcal{R}_t^C := \{ x \in \mathbb{R}^d, \exists \tau \in [0,t], \exists y_x \in S_{[0,\tau]}(x), y_x(\tau) \in C \}$$

Theoretical properties of the (backward) reachable sets have been studied by many authors using the viability theory (Refs: Cellina, Aubin, Frankowska, Cannarsa, Clarke, Colombo, Lowendal, Rifford, Wolenski, ....)
Optimization-based controller design

Minimum-time control problem:

$$\mathcal{T}(x) = \inf \{ t; \ y_x(t) \in C, \ y_x \in S_{[0,t]}(x) \}.$$
Optimization-based controller design

- **Minimum-time control problem:**

  \[
  \mathcal{T}(x) = \inf \{ t; \ y_x(t) \in \mathcal{C}, \ y_x \in S_{[0,t]}(x) \}.
  \]

- The sublevels of the minimum function $\mathcal{T}$ correspond to the Capture Basins of the target $\mathcal{C}$:

  \[
  \mathcal{R}_C^{\#} = \{ x \in \mathbb{R}^d \mid \mathcal{T}(x) = t \}, \quad (3)
  \]

  \[
  \mathcal{R}_C^t = \{ x \in \mathbb{R}^d \mid \mathcal{T}(x) \leq t \}. \quad (4)
  \]
Zermelo problem
Consider the function $\Phi(x) = d_C(x)$.

- A Mayer’s problem:
  $$V(x, t) = \inf_{y_x \in S_{[0,t]}(x)} \Phi(y_x(t))$$

- A Mayer’s problem with extended trajectories
  $$\hat{V}(x, t) = \inf_{y_x \in \hat{S}_{[0,t]}(x)} \Phi(y_x(t))$$

where $\hat{S}_{[0,t]}(x)$ is the set of trajectories satisfying:

$$\dot{y}(s) = \lambda(s)f(y(s), \alpha(s)), \; y(0) = x,$$

with $\lambda(s) \in [0, 1]$ a.e is a new control variable.
Level set approach

- $\bar{V}$ and $\hat{\bar{V}}$ are Lipschitz continuous functions
Level set approach

- \( V \) and \( \hat{V} \) are Lipschitz continuous functions
- For every \( t \geq 0 \),

\[
\mathcal{R}_{\#t}^c = \{ x \in \mathbb{R}^d; V(x, t) \leq 0 \}; \\
\mathcal{R}_t^c = \{ x \in \mathbb{R}^d; \hat{V}(x, t) \leq 0 \}
\]
Level set approach

- $V$ and $\hat{V}$ are Lipschitz continuous functions
- For every $t \geq 0$,

$$
\mathcal{R}_{ct} = \{ x \in \mathbb{R}^d ; V(x, t) \leq 0 \};
\mathcal{R}_{t} = \{ x \in \mathbb{R}^d ; \hat{V}(x, t) \leq 0 \}
$$

- The minimum time function $T : \mathbb{R}^d \rightarrow \mathbb{R}^+ \cup \{+\infty\}$ is lsc. Moreover, we have:

$$
T(x) = \inf\{ t \geq 0 ; V(x, t) \leq 0 \} = \min\{ t \geq 0 ; \hat{V}(x, t) \leq 0 \}.
$$
Level set approach

- \( V \) and \( \hat{V} \) are Lipschitz continuous functions
- For every \( t \geq 0 \),
  \[
  \mathcal{R}_C^t = \{ x \in \mathbb{R}^d ; V(x, t) \leq 0 \}; \\
  \mathcal{T}_t^C = \{ x \in \mathbb{R}^d ; \hat{V}(x, t) \leq 0 \}
  \]
- The minimum time function \( T : \mathbb{R}^d \to \mathbb{R}^+ \cup \{ +\infty \} \) is lsc. Moreover, we have:
  \[
  T(x) = \inf\{ t \geq 0 ; V(x, t) \leq 0 \} = \min\{ t \geq 0 ; \hat{V}(x, t) \leq 0 \}.
  \]
- \( \Phi \) can be any function satisfying
  \[
  \Phi(x) \leq 0 \iff x \in C.
  \]
Van der Pol Problem:
\[
\begin{align*}
\dot{y}_1(t) &= y_2 \\
\dot{y}_2(t) &= -y_1 + y_2(1 - y_1^2) + a(t) \\
a(t) &\in [-1, 1]
\end{align*}
\]

\(\Phi(y) := \|y\| - r_0 = d_C(y)\)

The HJB equation to be solved for this problem is:
\[
-\partial_t \psi(x, t) + \begin{pmatrix} -x_2 \\ x_1 + x_2(1 - x_1^2) \end{pmatrix} \cdot D_x \psi(x, t) + |\partial_{x_2} \psi(x, t)| = 0
\]
\(\psi(x, 0) = d_C(x).\)
Value function and optimal trajectories for a control problem with supremum cost function and state constraints

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joint work with: A. Assellaou, & O. Bokanowski, & A. Desilles
Consider the following state constrained control problem:

\[
\vartheta(t, y) := \inf \left\{ \max_{\theta \in [0, t]} \Phi(y_u^u(\theta)) \mid u \in \mathcal{U}, \ y_u^u(s) \in \mathcal{K}, \ s \in [0, t] \right\}
\]  

(1)

where \( y_u^u \) denotes the solution of the controlled differential system:

\[
\begin{align*}
\dot{y}(s) &:= f(y(s), u(s)), \quad \text{a.e } s \in [0, t], \\
y(0) &:= y,
\end{align*}
\]

\( \mathcal{K} \) is a closed set of \( \mathbb{R}^d \) and \( \mathcal{U} \) is a set of admissible control inputs.
Hamilton-Jacobi approach

- How can we handle correctly the state constraints?
- What boundary conditions should be considered for the HJB equation?
- Trajectory reconstruction and feedback control law.
- When $\Phi \geq 0$, we know that

$$
\left( \int_0^t \Phi(y^u(s))^{2p} \, ds \right)^{\frac{1}{2p}} \rightarrow \max_{s \in [0,t]} \Phi(y^u(s)), \quad \text{as } p \rightarrow +\infty.
$$

So, it is possible to approximate the maximum running cost problem by a Bolza problem. Does this approximation work well in practice?
Outline

1 Motivation: Abort landing problem in presence of "Wind Shear"

2 Optimal control problem with maximum-cost and state constraints

3 Optimal trajectories

4 Numerical simulations
1. Motivation: Abort landing problem in presence of "Wind Shear"

2. Optimal control problem with maximum-cost and state constraints

3. Optimal trajectories

4. Numerical simulations
Abort landing problem in presence of windshear


Consider the flight motion of an aircraft in a vertical plane:

\[
\begin{align*}
\dot{x} &= V \cos \gamma + w_x \\
\dot{h} &= V \sin \gamma + w_h \\
\dot{V} &= \frac{F_T}{m} \cos(\alpha + \delta) - \frac{F_D}{m} - g \sin \gamma - (\dot{w}_x \cos \gamma + \dot{w}_h \sin \gamma) \\
\dot{\gamma} &= \frac{1}{V} \left( \frac{F_T}{m} \sin(\alpha + \delta) + \frac{F_L}{m} - g \cos \gamma + (\dot{w}_x \sin \gamma - \dot{w}_h \cos \gamma) \right)
\end{align*}
\]

where

\[
\begin{align*}
\dot{w}_x &= \frac{\partial w_x}{\partial x} (V \cos \gamma + w_x) + \frac{\partial w_x}{\partial h} (V \sin \gamma + w_h) \\
\dot{w}_h &= \frac{\partial w_h}{\partial x} (V \cos \gamma + w_x) + \frac{\partial w_h}{\partial h} (V \sin \gamma + w_h)
\end{align*}
\]

and

- $F_T := F_T(V)$ is the thrust force
- $F_D := F_D(V, \alpha)$ and $F_L := F_L(V, \alpha)$ are the drag and lift forces
- $w_x := w_x(x)$ and $w_h := w_h(x, h)$ are the wind components
- $m$, $g$, and $\delta$ are constants.
Controlled system

- Consider the state $\mathbf{y}(.) = (x(.), h(.), V(.), \gamma(.), \alpha(.))$.
- The control variable $\mathbf{u}$ is the angular speed of the angle of attack $\alpha$.
- Let $T$ be a fixed time horizon and let $\mathcal{U}$ be the set of admissible controls

$$\mathcal{U} := \left\{ \mathbf{u} : (0, T) \to \mathbb{R}, \text{ measurable, } \mathbf{u}(t) \in U \text{ a.e} \right\}$$

where $U$ is a compact set.
- The controlled dynamics in this case is:

$$\begin{cases} 
\dot{x} = V \cos \gamma + w_x, \\
\dot{h} = V \sin \gamma + w_h, \\
\dot{V} = \frac{F_T}{m} \cos(\alpha + \delta) - \frac{F_D}{m} - g \sin \gamma - (\dot{w}_x \cos \gamma + \dot{w}_h \sin \gamma), \\
\dot{\gamma} = \frac{1}{V} \left( \frac{F_T}{m} \sin(\alpha + \delta) + \frac{F_l}{m} - g \cos \gamma + (\dot{w}_x \sin \gamma - \dot{w}_h \cos \gamma) \right), \\
\dot{\alpha} = \mathbf{u}.
\end{cases}$$
Formulation of the optimal control problem

- **Aim**: Maximize the minimal altitude over a time interval:

\[
\min_{\theta \in [0, t]} h(\theta)
\]

while the aircraft stays in a given domain \( \mathcal{K} \).

- Consider the following optimal control problem:

\[
(P) : \quad \vartheta(t, y) = \inf \left\{ \max_{\theta \in [0, t]} \Phi(y^u_y(\theta)), \quad u \in \mathcal{U}, \text{ and } y^u_y(s) \in \mathcal{K}, \; \forall s \in [0, t] \right\}
\]

where \( \Phi(y^u_y(.)) = H_r - h(.) \), \( H_r \) being a reference altitude, and \( \mathcal{K} \) is a set of state constraints.
Formulation of the optimal control problem

- **Aim**: Maximize the minimal altitude over a time interval:

  \[ \min_{\theta \in [0,t]} h(\theta) \]

  while the aircraft stays in a given domain \( \mathcal{K} \).

- Consider the following optimal control problem:

  \[ (P) : \vartheta(t, y) = \inf \left\{ \max_{\theta \in [0,t]} \Phi(y^u_\gamma(\theta)), |u| \in \mathcal{U}, \text{ and } y^u_\gamma(s) \in \mathcal{K}, \forall s \in [0, t] \right\} \]

  where \( \Phi(y^u_\gamma(.)) = H_r - h(.) \), \( H_r \) being a reference altitude, and \( \mathcal{K} \) is a set of state constraints.
Outline

1. Motivation: Abort landing problem in presence of "Wind Shear"

2. Optimal control problem with maximum-cost and state constraints

3. Optimal trajectories

4. Numerical simulations
A general setting

- For a given non-empty compact subset $U$ of $\mathbb{R}^k$ and a finite time $T > 0$, define the set of admissible control to be,

$$
U := \left\{ u : (0, T) \rightarrow \mathbb{R}^k, \text{ measurable, } u(t) \in U \text{ a.e} \right\}.
$$

- Consider the following control system:

$$
\begin{align*}
\dot{y}(s) &:= f(y(s), u(s)), \quad \text{a.e } s \in [0, T], \\
y(0) &:= y,
\end{align*}
$$

where $u \in U$ and the function $f$ is defined and continuous on $\mathbb{R}^d \times U$ and that it is Lipschitz continuous w.r.t $y$,

$$
\begin{align*}
(i) & \quad f : \mathbb{R}^d \times U \rightarrow \mathbb{R}^d \text{ is continuous,} \\
(ii) & \quad \exists L > 0 \text{ s.t. } \forall (y_1, y_2) \in \mathbb{R}^d \times \mathbb{R}^d, \forall u \in U, |f(y_1, u) - f(y_2, u)| \leq L(|y_1 - y_2|).
\end{align*}
$$

- The corresponding set of feasible trajectories:

$$
S_{[0, T]}(y) := \{ y \in W^{1,1}(0, T; \mathbb{R}^d), \text{ y satisfies (2) for some } u \in U \},
$$
A general setting

For a given non-empty compact subset $U$ of $\mathbb{R}^k$ and a finite time $T > 0$, define the set of admissible control to be,

$$\mathcal{U} := \left\{ u : (0, T) \rightarrow \mathbb{R}^k, \text{ measurable, } u(t) \in U \text{ a.e} \right\}.$$ 

Consider the following control system:

$$\begin{cases} \dot{y}(s) := f(y(s), u(s)), & \text{a.e } s \in [0, T], \\ y(0) := y, \end{cases}$$

where $u \in \mathcal{U}$ and the function $f$ is defined and continuous on $\mathbb{R}^d \times U$ and that it is Lipschitz continuous w.r.t $y$,

$$\begin{cases} (i) f : \mathbb{R}^d \times U \rightarrow \mathbb{R}^d \text{ is continuous,} \\ (ii) \exists L > 0 \text{ s.t. } \forall (y_1, y_2) \in \mathbb{R}^d \times \mathbb{R}^d, \forall u \in U, |f(y_1, u) - f(y_2, u)| \leq L(|y_1 - y_2|). \end{cases}$$

The corresponding set of feasible trajectories:

$$S_{[0, T]}(y) := \{ y \in W^{1,1}(0, T; \mathbb{R}^d), \ y \text{ satisfies } (2) \text{ for some } u \in \mathcal{U}\},$$
State constrained control problem with maximum cost

Consider the following state constrained control problem:

\[
\vartheta(t, y) := \inf \left\{ \max_{\theta \in [0, t]} \Phi(y^u_y(\theta)) \mid u \in U, \ y^u_y(s) \in \mathcal{K}, \ s \in [0, t] \right\} \tag{3}
\]

- the cost function \(\Phi(\cdot)\) is assumed to be Lipschitz continuous and \(\mathcal{K}\) is a closed set of \(\mathbb{R}^d\).
- For every \(y \in \mathbb{R}^d\), the set \(f(y, U) = \{f(y,u), \ u \in U\}\) is assumed to be convex.
- The function \(\vartheta\) is lower semicontinuous (lsc) on \(\mathcal{K}\), \(\vartheta \equiv +\infty\) outside \(\mathcal{K}\).
State constrained control problem with maximum cost

Consider the following state constrained control problem:

\[
\vartheta(t, y) := \inf \left\{ \max_{\theta \in [0, t]} \Phi(y^u_y(\theta)) \mid u \in U, \ y^u_y(s) \in \mathcal{K}, \ s \in [0, t] \right\}
\]  

(3)

- the cost function \(\Phi(\cdot)\) is assumed to be Lipschitz continuous and \(\mathcal{K}\) is a closed set of \(\mathbb{R}^d\).

- For every \(y \in \mathbb{R}^d\), the set \(f(y, U) = \{f(y, u), \ u \in U\}\) is assumed to be convex.

- The function \(\vartheta\) is lower semicontinuous (lsc) on \(\mathcal{K}\), \(\vartheta \equiv +\infty\) outside \(\mathcal{K}\).
State constrained control problem with maximum cost

Consider the following state constrained control problem:

\[
\vartheta(t, y) := \inf \left\{ \max_{\theta \in [0, t]} \Phi(y^u_y(\theta)) \mid u \in \mathcal{U}, \; y^u(s) \in \mathcal{K}, \; s \in [0, t] \right\}
\]  

(3)

- the cost function $\Phi(\cdot)$ is assumed to be Lipschitz continuous and $\mathcal{K}$ is a closed set of $\mathbb{R}^d$.

- For every $y \in \mathbb{R}^d$, the set $f(y, U) = \{f(y, u), \; u \in U\}$ is assumed to be convex.

- The function $\vartheta$ is lower semicontinuous (lsc) on $\mathcal{K}$, $\vartheta \equiv +\infty$ outside $\mathcal{K}$.
Some references

- **Maximum cost problems without state constraints** ($\mathcal{K} = \mathbb{R}^d$): Barron-Ishii (99)

- **Bolza or Mayer problems with state constraints**: Soner (86), Rampazzo-Vinter (89), Frankowska-Vinter (00), Motta (95), Cardaliaguet-Quincampoix-Saint-Pierre (97), Altarovici-Bokanowski-HZ (13), Hermosilla-HZ (15)

- **Maximum cost problems with state constraints**: Quincampoix-Serea (02), Bokanowski-Picarelli-HZ (13), Assellaou-Bokanowski-Desilles-HZ (CDC’16), Assellaou-Bokanowski-Desilles-HZ (preprint’16)
An auxiliary control problem

Let \( g \) be a Lipschitz continuous function such that:

\[
\forall y \in \mathcal{K}, \quad g(y) \leq 0 \iff y \in \mathcal{K}.
\]  

Consider the following auxiliary control problem:

\[
w(t, y, z) := \inf_{y \in S_{[0,t]}(y)} \max_{\theta \in [0,t]} \left( \Phi(y(\theta)) - z - g(y(\theta)) \right),
\]

where \( a \lor b = \max(a, b) \).

Theorem

Let \( (t, y, z) \in [0, T] \times \mathcal{K} \times \mathbb{R} \). The following assertions hold:

(i) \( \vartheta(t, y) - z \leq 0 \iff w(t, y, z) \leq 0 \),

(ii) \( \vartheta(t, y) = \min \left\{ z \in \mathbb{R} \mid w(t, y, z) \leq 0 \right\} \).
Define the following Hamiltonian as:

\[ H(y, p) := \max_{u \in U} (-f(y, u) \cdot p) \quad \forall y, p \in \mathbb{R}^d. \]

**Proposition**

The value function \( w \) is the unique Lipschitz continuous viscosity solution of the following Hamilton-Jacobi-Bellman (HJB) equation:

\[
\min \left( \partial_t w(t, y, z) + H(y, \nabla_y w), \ w(t, y, z) - \psi(y, z) \right) = 0 \quad ]0, T[ \times \mathbb{R}^d \times \mathbb{R},
\]

\[ w(0, y, z) = \psi(y, z), \quad \mathbb{R}^d \times \mathbb{R}, \]

where \( \psi(y, z) = (\Phi(y) - z) \vee g(y) \).
A particular choice of function $g$

Let $\eta > 0$ and define the following extended set $wK$:

$$K_\eta : = K + B(0, \eta).$$

$g(y) := d_K(y)$ the signed distance to $K$.

Consider the following auxiliary control problem:

$$w(t, y, z) : = \inf_{y \in S_{[0, t]}(y)} \left[ \max_{\theta \in [0, t]} \left( \Phi(y(\theta)) - z \right) \vee g(y(\theta)) \right] \wedge \eta,$$

where $a \wedge b = \min(a, b)$.

Theorem

Let $(t, y, z) \in [0, T] \times K \times \mathbb{R}$. The following assertions hold:

(i) $\varphi(t, y) - z \leq 0 \Leftrightarrow w(t, y, z) \leq 0,$

(ii) $\varphi(t, y) = \min \left\{ z \in \mathbb{R} : w(t, y, z) \leq 0 \right\}.$
A particular choice of function $g$

**Theorem**

The function $w$ is the unique Lipschitz continuous viscosity solution of the following HJB equation:

\[
\min \left( \partial_t w(t, y, z) + H(y, \nabla_y w), \ w(t, y, z) - \psi_\eta(y, z) \right) = 0 \ \ \ \ \ ]0, T[ \times \mathcal{K}_\eta \times \mathbb{R},
\]

\[
w(0, y, z) = \psi_\eta(y, z), \quad \mathcal{K}_\eta \times \mathbb{R},
\]

\[
w(t, y, z) = \eta, \quad y \notin \mathcal{K}_\eta,
\]

where $\psi_\eta(y, z) = \left[ (\Phi(y) - z) \lor g(y) \right] \land \eta$. 

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Define the exit time function:

\[
\mathcal{T}(y, z) := \sup \{ t \in [0, T] \mid \vartheta(t, y) \leq z \} \\
= \sup \{ t \in [0, T] \mid w(t, y, z) \leq 0 \}
\]

Link with viability theory:

(i) \( \mathcal{T} \) is the exit time function for \( \text{Epi}(\Phi) \cap (K \times \mathbb{R}^d) \),
(ii) \( \mathcal{T}(y, z) = t \Rightarrow w(t, y, z) = 0 \),
(iii) \( \vartheta(t, y) = \inf \{ z \mid \mathcal{T}(y, z) \geq t \} \).
Define the exit time function:

\[
\mathcal{T}(y, z) := \sup\{ t \in [0, T] \mid \vartheta(t, y) \leq z \}
\]

\[
= \sup\{ t \in [0, T] \mid w(t, y, z) \leq 0 \}
\]

Link with viability theory:

(i) \( \mathcal{T} \) is the exit time function for \( \mathcal{Epi}(\Phi) \cap (\mathcal{K} \times \mathbb{R}^d) \),

(ii) \( \mathcal{T}(y, z) = t \Rightarrow w(t, y, z) = 0 \),

(iii) \( \vartheta(t, y) = \inf \{ z \mid \mathcal{T}(y, z) \geq t \} \).
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1 Motivation: Abort landing problem in presence of "Wind Shear"

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Proposition

Let $y \in \mathcal{K}$ such that $\vartheta(T, y) < \infty$. Define $z^* := \vartheta(T, y)$.

- Let $\hat{y}^* = (y^*, z^*)$ be the optimal trajectory for the auxiliary control problem associated with the initial point $(y, z^*) \in \mathcal{K} \times \mathbb{R}$. Then, the trajectory $y^*$ is optimal for the original control problem.

- Let $\hat{y}^* = (y^*, z^*)$ be an optimal trajectory for the exit time problem associated with the initial point $(y, z) \in \mathcal{K} \times \mathbb{R}$. Then, $\hat{y}^*$ is also optimal for the auxiliary control problem.
Reconstruction of optimal trajectories - Algorithm A.

➢ For $n \geq 1$, consider $(t_0 = 0, t_1, \ldots, t_{n-1}, t_n = T)$ a uniform partition of $[0, T]$ with $\Delta t = \frac{T}{n}$.

➢ Let $\{y^n(\cdot), z^n(\cdot)\}$ be a trajectory defined recursively on the intervals $(t_{i-1}, t_i]$, with $z^n(\cdot) := z = \vartheta(0, y)$ and $y^n(0) = y$.

➢ [Step 1] Knowing $y^n_k = y^n(t_k)$, choose the optimal control at $t_k$ s.t.:

$$u^n_k \in \arg \min_{u \in U} \left( w(t_k, y^n_k + \Delta t f(t_k, u), z) + \lambda_n C(u, \Delta t) \right).$$

➢ [Step 2] Define $u^n(t) := u^n_k$, $\forall t \in (t_k, t_{k+1}]$ and $y^n(t)$ on $(t_k, t_{k+1}]$ as the solution of

$$\dot{y}(t) := f(y(t), u^n(t)) \text{ a.e } t \in (t_k, t_{k+1}],$$

with initial condition $y^n(t_k)$ at $t_k$ and $z^n(\cdot) := z$. 
Reconstruction of optimal trajectories - **Algorithm A.**

- For $n \geq 1$, consider $(t_0 = 0, t_1, ..., t_{n-1}, t_n = T)$ a uniform partition of $[0, T]$ with $\Delta t = \frac{T}{n}$.
- Let $\{y^n(\cdot), z^n(\cdot)\}$ be a trajectory defined recursively on the intervals $(t_{i-1}, t_i]$, with $z^n(\cdot) := z = \vartheta(0, y)$ and $y^n(0) = y$.

- **[Step 1]** Knowing $y^n_k = y^n(t_k)$, choose the optimal control at $t_k$ s.t.:

$$u^n_k \in \arg \min_{u \in U} \left( w(t_k, y^n_k + \Delta t f(t, y^n_k, u), z) + \lambda_n C(u, \Delta t) \right).$$

- **[Step 2]** Define $u^n(t) := u^n_k$, $\forall t \in (t_k, t_{k+1}]$ and $y^n(t)$ on $(t_k, t_{k+1}]$ as the solution of

$$\dot{y}(t) := f(y(t), u^n(t)) \text{ a.e } t \in (t_k, t_{k+1}],$$

with initial condition $y^n(t_k)$ at $t_k$ and $z^n(\cdot) := z$. 
Reconstruction of optimal trajectories - **Algorithm A.**

- For $n \geq 1$, consider $(t_0 = 0, t_1, ..., t_{n-1}, t_n = T)$ a uniform partition of $[0, T]$ with $\Delta t = \frac{T}{n}$.

- Let $\{y^n(\cdot), z^n(\cdot)\}$ be a trajectory defined recursively on the intervals $(t_{i-1}, t_i]$, with $z^n(\cdot) := z = \vartheta(0, y)$ and $y^n(0) = y$.

- **[Step 1]** Knowing $y^n_k = y^n(t_k)$, choose the optimal control at $t_k$ s.t.:

  \[
  u^n_k \in \arg \min_{u \in U} \left( w(t_k, y^n_k + \Delta t f_{\Delta t}(y^n_k, u), z) + \lambda C(u, \Delta t) \right). \]

- **[Step 2]** Define $u^n(t) := u^n_k$, $\forall t \in (t_k, t_{k+1}]$ and $y^n(t)$ on $(t_k, t_{k+1}]$ as the solution of

  \[
  \dot{y}(t) := f(y(t), u^n(t)) \quad \text{a.e } t \in (t_k, t_{k+1}],
  \]

  with initial condition $y^n(t_k)$ at $t_k$ and $z^n(\cdot) := z$. 

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Theorem

Let \( \{y^n(\cdot), z^n(\cdot), u^n(\cdot)\} \) be a sequence generated by algorithm A for \( n \geq 1 \). Then, the sequence of trajectories \( \{y^n(\cdot)\}_n \) has cluster points with respect to the uniform convergence topology. For any cluster point \( \bar{y}(\cdot) \) there exists a control law \( \bar{u}(\cdot) \) such that \( (\bar{y}(\cdot), \bar{z}(\cdot), \bar{u}(\cdot)) \) is optimal for the auxiliary control problem.

Let \( w^{\Delta} \) be a numerical approximate solution such that,

\[
|w^{\Delta}(t, y, z) - w(t, y, z)| \leq E_1(\Delta t, \Delta y),
\]

where \( E_1(\Delta t, \Delta y) \to 0 \) as \( \Delta t, \Delta y \to 0 \).

Let \( \{Y^n(\cdot), u^n(\cdot)\} \) be the sequence generated by the algorithm A with \( w^{\Delta} \).

Then, \( (Y^n) \) converges to an optimal trajectory for the auxiliary control problem.
Theorem

Let \( \{y^n(\cdot), z^n(\cdot), u^n(\cdot)\} \) be a sequence generated by algorithm A for \( n \geq 1 \). Then, the sequence of trajectories \( \{y^n(\cdot)\}_n \) has cluster points with respect to the uniform convergence topology. For any cluster point \( \bar{y}(\cdot) \) there exists a control law \( \bar{u}(\cdot) \) such that \( (\bar{y}(\cdot), \bar{z}(\cdot), \bar{u}(\cdot)) \) is optimal for the auxiliary control problem.

Let \( w^\Delta \) be a numerical approximate solution such that,

\[
|w^\Delta(t, y, z) - w(t, y, z)| \leq E_1(\Delta t, \Delta y),
\]

where \( E_1(\Delta t, \Delta y) \rightarrow 0 \) as \( \Delta t, \Delta y \rightarrow 0 \).

Let \( \{Y^n(\cdot), u^n(\cdot)\} \) be the sequence generated by the algorithm A with \( w^\Delta \).

Then, \( (Y^n)_n \) converges to an optimal trajectory for the auxiliary control problem.
Theorem

Let \( \{y^n(\cdot), z^n(\cdot), u^n(\cdot)\} \) be a sequence generated by algorithm A for \( n \geq 1 \). Then, the sequence of trajectories \( \{y^n(\cdot)\}_n \) has cluster points with respect to the uniform convergence topology. For any cluster point \( \bar{y}(\cdot) \) there exists a control law \( \bar{u}(\cdot) \) such that \( (\bar{y}(\cdot), \bar{z}(\cdot), \bar{u}(\cdot)) \) is optimal for the auxiliary control problem.

Let \( w^\Delta \) be a numerical approximate solution such that,

\[
|w^\Delta(t, y, z) - w(t, y, z)| \leq E_1(\Delta t, \Delta y),
\]

where \( E_1(\Delta t, \Delta y) \to 0 \) as \( \Delta t, \Delta y \to 0 \).

Let \( \{Y^n(\cdot), u^n(\cdot)\} \) be the sequence generated by the algorithm A with \( w^\Delta \).

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Reconstruction of optimal trajectories using the exit time - Algorithm B.

For \( n \geq 1 \), consider \((t_0 = 0, t_1, \ldots, t_{n-1}, t_n = T)\) a uniform partition of \([0, T]\) with \( \Delta t = \frac{T}{n} \).

Let \( \{y^n(\cdot), z^n(\cdot)\} \) be a trajectory defined recursively on the intervals \((t_{i-1}, t_i]\), with \( z^n(\cdot) := z = \vartheta(T, y) \) and \( y^n(t_0) = y \).

[Step 1] Knowing \( y^n_k = y^n(t_k) \), choose the optimal control at \( t_k \) s.t.:

\[
\begin{align*}
    u^n_k \in \arg \max_{u \in U} \left\{ \left( T(y^n(t_k) + \Delta t f_{\Delta t}(y^n(t_k), u), z) + \Delta t \right) \wedge T \right\},
\end{align*}
\]

[Step 2] Define \( u^n(t) := u^n_k \), \( \forall t \in (t_k, t_{k+1}] \) and \( y^n(t) \) on \((t_k, t_{k+1}]\) as the solution of

\[
\begin{align*}
    \dot{y}(t) := f(y(t), u^n(t)) \ a.e \ t \in (t_k, t_{k+1}],
\end{align*}
\]

with initial condition \( y^n(t_k) \) at \( t_k \) and \( z^n(\cdot) := z \), with initial condition \( y^n(t_k) \) at \( t_k \) and \( z^n(\cdot) := z \).
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[Step 1] Knowing $y^n_k = y^n(t_k)$, choose the optimal control at $t_k$ s.t.:

$$u^n_k \in \arg \max_{u \in U} \left\{ \left( T(y^n(t_k) + \Delta t f_{\Delta t}(y^n(t_k), u), z) + \Delta t \right) \wedge T \right\},$$

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Reconstruction of optimal trajectories using the exit time

- Algorithm B.

- For $n \geq 1$, consider $(t_0 = 0, t_1, \ldots, t_{n-1}, t_n = T)$ a uniform partition of $[0, T]$ with $\Delta t = \frac{T}{n}$.

- Let $\{y^n(\cdot), z^n(\cdot)\}$ be a trajectory defined recursively on the intervals $(t_{i-1}, t_i]$, with $z^n(\cdot) := z = \theta(T, y)$ and $y^n(t_0) = y$.

- [Step 1] Knowing $y^n_k = y^n(t_k)$, choose the optimal control at $t_k$ s.t.:

$$u^n_k \in \arg \max_{u \in U} \left\{ \left( T(y^n(t_k) + \Delta t f_{\Delta t}(y^n(t_k), u), z) + \Delta t \right) \wedge T \right\},$$

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with initial condition $y^n(t_k)$ at $t_k$ and $z^n(\cdot) := z$. with initial condition $y^n(t_k)$ at $t_k$ and $z^n(\cdot) := z$. 
Outline

1. Motivation: Abort landing problem in presence of "Wind Shear"

2. Optimal control problem with maximum-cost and state constraints

3. Optimal trajectories

4. Numerical simulations
Numerical schemes

- **Finite Difference scheme**

\[
W_{I,j}^{n+1} = \max \left( W_{I,j}^n + \Delta t \mathcal{H}(y_I, D^+ W^n(y_I, z_j), D^- W^n(y_I, z_j)), \Psi_{I,j} \right)
\]

\[
W_{I,j}^N = \Psi_{I,j},
\]

- **Semi Lagrangian scheme**

\[
W_{I,j}^{n+1} = \min_{a \in U} \left( W^n(y_I + f(y_I, a) \Delta t, z_j) \right) \lor \Psi_{I,j}
\]

\[
W_{I,j}^N = \Psi_{I,j}
\]

- Same error estimates for both schemes under adequate CFL conditions.
- Application on the Wind Shear problem (Boeing 727 aircraft model data)
- Simulations on a grid with \( N_G = 40^3 \times 20^2 \times 10 \) nodes (where 30 is the number of points per axis for the first three components, namely, \( x, h \) and \( v \), 20 is the number of the points for the angles \( \gamma \) and \( \alpha \) an 10 is the number of points for the additional variable \( z \))
Approximation by Bolza problems

\[
\left( \int_0^t \Phi(y^u_y(s))^{2p} \, ds \right)^{\frac{1}{2p}} \rightarrow \max_{s \in [0,t]} \Phi(y^u_y(s)), \quad \text{as } p \rightarrow +\infty.
\]

<table>
<thead>
<tr>
<th>Criterion</th>
<th>Optimal cost value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bolza, 2p = 6</td>
<td>907.59</td>
</tr>
<tr>
<td>Bolza, 2p = 8</td>
<td>807.64</td>
</tr>
<tr>
<td>Bolza, 2p = 10</td>
<td>728.08</td>
</tr>
<tr>
<td>Bolza, 2p = 12</td>
<td>686.29</td>
</tr>
<tr>
<td>Maximum running cost</td>
<td>611.72</td>
</tr>
</tbody>
</table>
Numerical simulations

- Quite similar optimal trajectories by using either algorithms A and B. However, using the exit time function is much more cheaper (in term of data storage).
- The control variable enters linearly:

\[ f(y, u) = uf_0(y) + f_1(y). \]

The Hamiltonian has a simple form (assume \( u \in [-1, 1] \)):

\[ H(x, p) := -f_1(y) \cdot p + |f_0(y) \cdot p|. \]

However, the optimal control strategy may be of Bang-bang type or may include "singular arcs" when the gradient of the value function is close to 0.
Figure: Reconstruction of the state variables by adding a penalization term 
\( \lambda C(u, u_n) = |u - u_n| \), with \( \lambda = 0.0, 1.0 \) and 2.0.
... thank you for your attention.